THE NORM THEOREM FOR SEMISINGULAR QUADRATIC FORMS

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ABSTRACT. Let F be a field of characteristic 2. The aim of this paper is to give a complete proof of the norm theorem for singular F-quadratic forms which are not totally singular, i.e., we give necessary and sufficient conditions for which a normed irreducible polynomial of $F[x_1, \ldots, x_n]$ becomes a norm of such a quadratic form over the rational function field $F(x_1, \ldots, x_n)$. This completes partial results proved on this question in [9]. Combining the present work with the papers [1] and [8], we obtain the norm theorem for any type of quadratic forms in characteristic 2.

1. INTRODUCTION

Let F be an arbitrary field. Given a field extension K/F, a natural problem in the algebraic theory of quadratic forms consists of studying the behavior of F-quadratic forms after scalar extension to K. This problem first started by a result of Witt [11] characterizing irreducible polynomials p in one variable for which a given anisotropic F-quadratic form becomes isotropic over $F(p)$ (the function field of the affine hypersurface given by $p = 0$). Few years later, Knebusch studied the problem of metabolicity of F-bilinear forms over $F(p)$, where $p \in F[x_1, \ldots, x_n]$ is normed irreducible (normed means that the coefficient of the highest monomial occurring in p with respect to the lexicographical ordering is 1). To this end, he introduced in [6], his specialization theory for quadratic and bilinear forms. As a consequence, he proved an important result known as the norm theorem [6, Theorem 4.2]. The result states that, for a normed irreducible polynomial $p \in F[x_1, x_2, \ldots, x_n]$, an anisotropic bilinear form $\mathfrak{b}$ over F becomes metabolic over $F(p)$ if and only if p is a norm of $\mathfrak{b}$ over $F(x_1, \ldots, x_n)$, i.e., $\mathfrak{b}$ is isometric to $p\mathfrak{b}$ over $F(x_1, \ldots, x_n)$.

Obviously, Knebusch's norm theorem cited before extends to quadratic forms in characteristic different from 2. Concerning quadratic forms in characteristic 2, we distinguish between three types: nonsingular forms, semisingular forms and totally singular forms (see Section 2). Baeza extended the norm theorem to nonsingular forms [1]. His proof uses Knebusch's norm theorem for quadratic form in characteristic not 2 via a lifting argument from characteristic 2 to 0, which is based on the idea that any field of characteristic 2 can be viewed as the residue field of a complete discrete valued ring of characteristic 0. For the case of singular quadratic forms (i.e., semisingular or totally singular quadratic forms) in characteristic 2 the situation is more subtle. The main ingredient used in this case is the notion of quasi-hyperbolicity which is an extension of the notion of hyperbolicity. Recall that a singular form φ is quasi-hyperbolic if $\dim \varphi$ is even and $i_t(\varphi) \geq \dim \varphi/2$, where $i_t(\varphi)$ is the total index of φ (see Section 2). Note that a restrictive notion of quasi-hyperbolicity (i.e., $i_t(\varphi) = \dim \varphi/2$) was first used by Laghribi, but it turns out that these two notions coincide over the field $F(p)$ for which we are interested here (see Proposition 2.2(2.b)). Based on the notion of quasi-hyperbolicity, Laghribi [8] and independently Hoffmann [5] proved the norm theorem for totally singular quadratic forms. Later, Laghbribi and Mammone gave partial results on norm theorem for semisingular quadratic forms [9]. More precisely, one of the result given by them asserts that whenever an anisotropic semisingular quadratic form φ has a normed irreducible polynomial $p \in F[x_1, \ldots, x_n]$ as a

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norm, then it is quasi-hyperbolic over $F(p)$. The reverse implication has also been proved in their paper for the special case when the polynomial p is given by a quadratic form which represents 1.

From now on we consider F to be a field of characteristic 2. The aim of this paper is the following result which completes the proof of the norm theorem for semisingular quadratic forms.

Theorem 1.1. *Let φ be a nondefective semisingular quadratic form of dimension ≥ 3 over F, and let $p \in F[x_1, x_2, \dots, x_n]$ be a normed irreducible polynomial and $K = F(x_1, x_2, \dots, x_n)$. Then, the following two conditions are equivalent:*

(1) φ is quasi-hyperbolic over $F(p)$.
(2) p is a norm of φ_K.

As we said before implication (2) $\Rightarrow$ (1) has already been proved in [9]. So we will focus on the proof of the implication (1) $\Rightarrow$ (2) which will be done in two steps. First, we give the proof in the case of a normed irreducible polynomial in one variable. In this step we first prove the theorem for the polynomial $x^{2^n} + d$, and then generalize it to any one variable normed irreducible polynomial using Scharlau's transfer. In the second step, we will prove the theorem for a polynomial in more than one variable for which we will use an induction on the number of variables due to Knebusch. To proceed with the induction we will need the following proposition.

Proposition 1.2. *Let φ be a nondefective semisingular quadratic form of dimension ≥ 3 over F, $f \in F[x_1, \dots, x_n]$ and $K = F(x_1, \dots, x_n)$. Let p be a normed irreducible polynomial which divides f with an odd power. If f is a norm of φ_K, then p is also a norm of φ_K.*

For a general polynomial (not necessarily irreducible), we will use Theorem 1.1 to get the following norm criteria.

Corollary 1.3. *Let φ be a nondefective semisingular quadratic form and $q \in F[x_1, \dots, x_n]$ such that $q = cp_1^{\epsilon_1} \dots p_r^{\epsilon_r}$ with $c \in F^* := F \setminus \{0\}$, $\epsilon_i \in \mathbb{N}_0$ and $p_i \in F[x_1, \dots, x_n]$ normed irreducible polynomial for any $1 \leq i \leq r$. Then, the following two conditions are equivalent:*

(1) q is a norm of φ.
(2) c is a norm of φ and $\varphi_{F(p_i)}$ is quasi-hyperbolic when ϵ_i is odd.

2. Background

Recall that any quadratic form φ over F can be written up to isometry as follows:

(2.1) $$\varphi \simeq [a_1, b_1] \perp [a_2, b_2] \perp \dots \perp [a_r, b_r] \perp \langle c_1 \rangle \perp \dots \perp \langle c_s \rangle,$$

where $\simeq$ and $\perp$ denotes the isometry and orthogonal sum of quadratic forms, and $[a, b]$ (*resp.* $\langle a \rangle$) denotes the quadratic form $ax^2 + xy + by^2$ (*resp.* ax^2). Obviously, $\dim \varphi = 2r + s$ (the dimension of φ). The quadratic form $\langle c_1 \rangle \perp \dots \perp \langle c_s \rangle$ is unique up to isometry, we call it the quasilinear part of φ, and denote it by $\mathrm{ql}(\varphi)$. As in equation (2.1), the form φ is called:

- nonsingular (*resp.* singular) if $s = 0$ (*resp.* $s > 0$),
- totally singular if $r = 0$,
- semisingular if $r > 0$ and $s > 0$.

For $a_1, \dots, a_n \in F$, let $\langle a_1, \dots, a_n \rangle$ denote the totally singular quadratic form $\langle a_1 \rangle \perp \dots \perp \langle a_n \rangle$.

A quadratic form φ of underlying F-vector space V is called isotropic if there exists $v \in V \setminus \{0\}$ such that $\varphi(v) = 0$, otherwise φ is called anisotropic.

For an integer $n \geq 0$ and φ a quadratic form, we denote $n \times \varphi$ for the quadratic form $\underbrace{\varphi \perp \ldots \perp \varphi}_{n \text{ times}}$.

Recall that any quadratic form φ over F uniquely decomposes as follows:

$$\varphi \simeq \varphi_{an} \perp i \times [0,0] \perp j \times \langle 0 \rangle,$$

where φ_{an} is an anisotropic quadratic form. We call φ_{an} the anisotropic part of φ, and the integer i (*resp.* j) is called the Witt index (*resp.* the defect index) of φ. The integer $i+j$ is called the total index of φ. We denote i, j and $i+j$ by $i_W(\varphi)$, $i_d(\varphi)$ and $i_t(\varphi)$, respectively. The form φ is called nondefective if $i_d(\varphi) = 0$.

Two quadratic forms φ_1 and φ_2 are called Witt-equivalent, denoted $\varphi_1 \sim \varphi_2$, if there exists $m, n \in \mathbb{N}$ such that $\varphi_1 \perp m \times [0,0] \simeq \varphi_2 \perp n \times [0,0]$.

A quadratic form (φ, V) represents $\alpha \in F$ if there exists $v \in V$ such that $\varphi(v) = \alpha$. We denote by $D_F(\varphi)$ the set of values in F^* represented by φ.

We will need the following cancellation result:

Proposition 2.1. *(*[6, Proposition 1.2] *for (1),* [4, Lemma 2.6] *for (2)) Let φ_1, φ_2 be two quadratic forms (possibly singular). Suppose that one of the following conditions holds:*
(1) $\varphi_1 \perp \psi \simeq \varphi_2 \perp \psi$ for some nonsingular form ψ,
(2) $\varphi_1 \perp s \times \langle 0 \rangle \simeq \varphi_2 \perp s \times \langle 0 \rangle$ for some integer $s \geq 0$ and φ_1, φ_2 nondefective.
Then $\varphi_1 \simeq \varphi_2$.

For $q \in F[x_1, x_2, \ldots, x_n]$ an irreducible polynomial, let $F(q)$ be the field of fractions of the quotient ring $F[x_1, \ldots, x_n]/(q(x_1, \ldots, x_n))$. We call it the function field of q.

A scalar $\alpha \in F^* := F \setminus \{0\}$ is called a norm of φ if $\varphi \simeq \alpha\varphi$.

For a field extension K/F and φ an F-quadratic form, let φ_K denote the quadratic form $\varphi \otimes K$.

For $a_1, \ldots, a_n \in F^*$, let $\langle a_1, \ldots, a_n \rangle_b$ be the diagonal bilinear form defined by:

$$((x_1, \ldots, x_n), (y_1, \ldots, y_n)) \mapsto \sum_{i=1}^{n} a_i x_i y_i.$$

Let $W(F)$ (*resp.* $W_q(F)$) be the Witt ring of regular symmetric F-bilinear forms (*resp.* the Witt group of nonsingular F-quadratic forms). The group $W_q(F)$ is endowed with a $W(F)$-module structure as follows: To any regular symmetric F-bilinear form B on a vector space V and a nonsingular F-quadratic form φ on a vector space W, we associate a nonsingular quadratic form $B \otimes \varphi$ defined on $V \otimes_F W$ by:

$$B \otimes \varphi(v \otimes w) = B(v,v)\varphi(w) \text{ for any } (v,w) \in V \times W$$

and whose polar form is $B \otimes B_\varphi$, where B_φ is the polar form of φ.

All irreducible polynomials $p \in F[x_1, \ldots, x_n]$ that we will deal with are inseparable, i.e., $p \in F[x_1^2, \ldots, x_n^2]$ as statement (2) of the following proposition asserts:

Proposition 2.2. *Let φ be a semisingular quadratic form over F.*
(1) If φ is quasi-hyperbolic, then $\mathrm{ql}(\varphi)$ is also quasi-hyperbolic.
(2) If $\mathrm{ql}(\varphi)$ is anisotropic and $p \in F[x_1, \ldots, x_n]$ is irreducible such that $\varphi_{F(p)}$ is quasi-hyperbolic, then:

(2.a) p is inseparable.

(2.b) $i_d(\varphi_{F(p)}) = \frac{\dim \mathrm{ql}(\varphi)}{2}$ and $i_W(\varphi_{F(p)}) = \frac{\dim \varphi - \dim \mathrm{ql}(\varphi)}{2}$.

(2.c) If p is normed, then p is a norm of $\mathrm{ql}(\varphi)$.

(2.d) If L is the subfield of F generated over F^2 by the coefficients of p, then any nonzero scalar of L is a norm of $\mathrm{ql}(\varphi)$.

Proof. Let R be a nonsingular quadratic form such that $\varphi \simeq R \perp \mathrm{ql}(\varphi)$.
(1) We have $i_d(\varphi) = i_d(\mathrm{ql}(\varphi))$ by the uniqueness of the quasilinear part. Suppose that φ is quasi-hyperbolic. Then $i_t(\varphi) = i_W(\varphi) + i_d(\varphi) \geq \frac{\dim \varphi}{2}$. Hence, $\frac{\dim \varphi}{2} \leq \frac{\dim R}{2} + i_d(\varphi)$ because $i_W(\varphi) \leq \frac{\dim R}{2}$. Consequently, $\frac{\dim \mathrm{ql}(\varphi)}{2} \leq i_d(\varphi)$.

(2) Suppose that $\mathrm{ql}(\varphi)$ is anisotropic and $\varphi_{F(p)}$ is quasi-hyperbolic. By previous statement, $\mathrm{ql}(\varphi)_{F(p)}$ is quasi-hyperbolic. By [5, Theorem 6.10] and [8, Theorem 1.1], the polynomial p is inseparable, it is a norm of $\mathrm{ql}(\varphi)$ when p is normed, and $i_d(\mathrm{ql}(\varphi)_{F(p)}) = \frac{\dim \mathrm{ql}(\varphi)}{2}$. Now, the condition $\frac{\dim \varphi}{2} \leq i_W(\varphi_{F(p)}) + i_d(\varphi_{F(p)}) = i_W(\varphi_{F(p)}) + \frac{\dim \mathrm{ql}(\varphi)}{2}$, implies that $i_W(\varphi_{F(p)}) \geq \frac{\dim R}{2}$. Consequently, $i_W(\varphi_{F(p)}) = \frac{\dim \varphi - \dim \mathrm{ql}(\varphi)}{2}$. Hence, the statements $(2.a)$, $(2.b)$ and $(2.c)$. Statement $(2.d)$ is proved in [5, Theorem 6.7]. □

We will now give some results on transfer that will play a crucial role in our proofs.

Let K/F be a finite field extension and $s : K \to F$ be a nonzero F-linear map. For a quadratic form q on a K-vector space V, we associate $s_*(q)$ an F-quadratic form on V, viewed as an F-vector space, defined as follows:

$$s_*(q)(v) = s(q(v)) \text{ for all } v \in V.$$

Similarly if $\mathfrak{b}$ is a bilinear form on a K-vector space V, we associate $s_*(\mathfrak{b})$ an F-bilinear form on V defined as follows:

$$s_*(\mathfrak{b})(v, w) = s(\mathfrak{b}(v, w)) \text{ for all } v, w \in V.$$

Note that $\dim s_*(q) = [K : F] \dim q$, and $s_*(q_1 \perp q_2) \simeq s_*(q_1) \perp s_*(q_2)$ for any two K-quadratic (or K-bilinear) forms q_1 and q_2. Moreover, if q is a nonsingular (*resp.* totally singular) quadratic form, then the form $s_*(q)$ is also a nonsingular (*resp.* totally singular) quadratic form. We also have $s_*(\mathfrak{b})$ regular if $\mathfrak{b}$ is a regular bilinear form.

Another important result that we will use in the proofs is the Frobenius reciprocity which is given by the following proposition.

Proposition 2.3. *(*[3, Proposition 20.2] *Frobenius Reciprocity) Suppose that K/F is a finite extension and $s : K \to F$ is a nonzero F-linear map. Let q (resp. $\mathfrak{b}$) be a nonsingular quadratic form over F (resp. symmetric bilinear form over K). Then, there exists an isometry*

$$s_*(\mathfrak{b} \otimes q_K) \simeq s_*(\mathfrak{b}) \otimes q.$$

The Frobenius reciprocity also exists when $\mathfrak{b}$ is defined over F and q is defined over K. For more details we refer to [3, Section 20].

We recall a well known result on transfer:

Proposition 2.4. *(*[3, Lemmas 20.9, 20.12]*) Let $K = F(\alpha)$ be a simple extension of F of degree m. Let $s : K \to F$ be the F-linear map given by $s(1) = 1$ and $s(\alpha^i) = 0$ for all $1 \leq i \leq m-1$. Then, we have in $W(F)$:*

$$s_*(\langle 1 \rangle_b) = \begin{cases} \langle 1 \rangle_b & \text{if } m \text{ is odd,} \\ \langle 1, N_{K/F}(\alpha) \rangle_b & \text{if } m \text{ is even.} \end{cases}$$

$$s_*(\langle \alpha \rangle_b) = \begin{cases} \langle N_{K/F}(\alpha) \rangle_b & \text{if } m \text{ is odd,} \\ 0 & \text{if } m \text{ is even,} \end{cases}$$

where $N_{K/F}$ is the norm map of the extension K/F.

Corollary 2.5. *We keep the same notations and hypotheses as in Proposition 2.4. For any nonsingular F-quadratic form R, we have in $W_q(F)$:*

$$s_*(R) = \begin{cases} R & \text{if } m \text{ is odd,} \\ \langle 1, N_{K/F}(\alpha)\rangle_b \otimes R & \text{if } m \text{ is even.} \end{cases}$$

$$s_*(\alpha R) = \begin{cases} \langle N_{K/F}(\alpha)\rangle_b \otimes R & \text{if } m \text{ is odd,} \\ 0 & \text{if } m \text{ is even.} \end{cases}$$

Proof. We combine the Frobenius reciprocity with Proposition 2.4 and the facts that $R \simeq \langle 1\rangle_b \otimes R$ and $\alpha R \simeq \langle \alpha\rangle_b \otimes R$. $\square$

We will also need the following computation for totally singular forms:

Lemma 2.6. *Let $d \in F \setminus F^2$, $K = F(\sqrt{d})$ and ψ a totally singular F-quadratic form. For the F-linear map $s : F(\sqrt{d}) \to F$ given by $1 \mapsto 1$ and $\sqrt{d} \mapsto 0$, we have:*

$$s_*(\psi) \simeq \langle 1, d\rangle \otimes \psi \quad \text{and} \quad s_*(\sqrt{d}\psi) \simeq 2\dim\psi \times \langle 0\rangle.$$

Proof. Since $K^2 = F^2 + dF^2 \subset F$, it follows that $s(D_K(\sqrt{d}\psi)) = \{0\}$, which means that $s_*(\sqrt{d}\psi) \simeq 2\dim\psi \times \langle 0\rangle$ and $s_*(\psi) \simeq \psi \perp d\psi$. $\square$

3. Proof of Theorem 1.1 in the case of $p = x^{2^n} + d$

The starting point of our investigation on the norm theorem is the following result:

Proposition 3.1. *([9, Proposition 2.7]) Let φ be a semisingular F-quadratic form and $d \in F \setminus F^2$ such that $i_W(\varphi_{F(\sqrt{d})}) = \frac{\dim\varphi - \dim \mathrm{ql}(\varphi)}{2}$. Then, there exists a nonsingular F-quadratic form R such that $\varphi \sim R \perp \mathrm{ql}(\varphi)$ and $x^2 + d$ is a norm of R over $F(x)$.*

From this proposition we will derive the following corollary and then prove the same result for any extension of the form $F(\sqrt[2^n]{d})$.

Corollary 3.2. *If φ is a semisingular F-quadratic form and $d \in F \setminus F^2$ such that $i_W(\varphi_{F(\sqrt{d})}) = \frac{\dim\varphi - \dim \mathrm{ql}(\varphi)}{2}$, then $x^2 + d$ is a norm of $\varphi \perp d\mathrm{ql}(\varphi)$ over $F(x)$.*

Proof. Suppose that $i_W(\varphi_{F(\sqrt{d})}) = \frac{\dim\varphi - \dim \mathrm{ql}(\varphi)}{2}$. From Proposition 3.1, we have $\varphi \sim R \perp \mathrm{ql}(\varphi)$, where R is a nonsingular form over F which admits $x^2 + d$ as a norm. Thus, we have $\varphi \perp d\mathrm{ql}(\varphi) \sim R \perp \mathrm{ql}(\varphi) \perp d\mathrm{ql}(\varphi)$ and $R \simeq (x^2+d)R$. Since $\mathrm{ql}(\varphi) \perp d\mathrm{ql}(\varphi) \simeq \langle 1, d\rangle \otimes \mathrm{ql}(\varphi)$ and $x^2 + d$ is a norm of $\langle 1, d\rangle$, it follows that $x^2 + d$ is also a norm of $\mathrm{ql}(\varphi) \perp d\mathrm{ql}(\varphi)$.

Therefore, $\varphi \perp d\mathrm{ql}(\varphi) \sim (x^2 + d)(\varphi \perp d\mathrm{ql}(\varphi))$. Since the dimension of left and right hand sides are the same, it follows from Proposition 2.1(1) that $\varphi \perp d\mathrm{ql}(\varphi) \simeq (x^2 + d)(\varphi \perp d\mathrm{ql}(\varphi))$. $\square$

Proposition 3.3. *Let φ be a semisingular F-quadratic form and $d \in F$ such that $x^{2^n} + d$ is irreducible over F and $i_W(\varphi_{F(\sqrt[2^n]{d})}) = \frac{\dim\varphi - \dim \mathrm{ql}(\varphi)}{2}$. Then, $x^{2^n} + d$ is a norm of $\varphi \perp d\mathrm{ql}(\varphi)$ over $F(x)$.*

Proof. We proceed by induction on n. For $n = 1$ the proposition is nothing but the previous corollary.

Suppose $n \geq 2$, and the proposition is true for $n-1$. Let $L = F(\sqrt{d})$ and $\varphi \simeq R \perp \mathrm{ql}(\varphi)$ be a semisingular F-quadratic form such that $i_W(\varphi_{F(\sqrt[2^n]{d})}) = \frac{\dim\varphi - \dim \mathrm{ql}(\varphi)}{2}$. We consider φ over the field L. Since $F(\sqrt[2^n]{d}) = L\left(\sqrt[2^{n-1}]{\sqrt{d}}\right)$ and

$$i_W\left((R \perp \mathrm{ql}(\varphi))_{L\left(\sqrt[2^{n-1}]{\sqrt{d}}\right)}\right) = \frac{\dim\varphi - \dim \mathrm{ql}(\varphi)}{2},$$

it follows from induction hypothesis that we get over $L(x)$

$$\varphi \perp \sqrt{d}\mathrm{ql}(\varphi) \simeq (x^{2^{n-1}} + \sqrt{d})(\varphi \perp \sqrt{d}\mathrm{ql}(\varphi)). \tag{3.1}$$

Note that $L(x) = F(x)(x^{2^{n-1}} + \sqrt{d})$. Now, to descent the previous isometry to $F(x)$ we will use Scharlau's transfer and Frobenius reciprocity for the $F(x)$-linear map $s : L(x) \to F(x)$ given by:

$$1 \mapsto 1 \text{ and } x^{2^{n-1}} + \sqrt{d} \mapsto 0.$$

Using the isometry of totally singular forms $\langle 1, \sqrt{d}\rangle \simeq \langle 1, x^{2^{n-1}} + \sqrt{d}\rangle$, it follows that

$$\mathrm{ql}(\varphi) \perp \sqrt{d}\mathrm{ql}(\varphi) \simeq \mathrm{ql}(\varphi) \perp (x^{2^{n-1}} + \sqrt{d})\mathrm{ql}(\varphi).$$

Hence, equation (3.1) becomes

$$R \perp \mathrm{ql}(\varphi) \perp (x^{2^{n-1}} + \sqrt{d})\mathrm{ql}(\varphi) \simeq (x^{2^{n-1}} + \sqrt{d})R \perp \mathrm{ql}(\varphi) \perp (x^{2^{n-1}} + \sqrt{d})\mathrm{ql}(\varphi). \tag{3.2}$$

Moreover, as $N_{L(x)/F(x)}(x^{2^{n-1}} + \sqrt{d}) = x^{2^n} + d$, it follows from Corollary 2.5 and Lemma 2.6

$$s_*(R) \sim \langle 1, x^{2^n} + d\rangle_b \otimes R,$$

$$s_*(\langle x^{2^{n-1}} + \sqrt{d}\rangle_b \otimes R) \sim 0,$$

$$s_*(\mathrm{ql}(\varphi)) \simeq \mathrm{ql}(\varphi) \perp (x^{2^n} + d)\mathrm{ql}(\varphi),$$

$$s_*((x^{2^{n-1}} + \sqrt{d})\mathrm{ql}(\varphi)) \simeq 2\dim\mathrm{ql}(\varphi) \times \langle 0\rangle.$$

Now by applying s_* to equation (3.2), we get

$$\begin{aligned} R \perp (x^{2^n} + d)R \perp \langle 1, x^{2^n} + d\rangle \otimes \mathrm{ql}(\varphi) \perp 2\dim\mathrm{ql}(\varphi) \times \langle 0\rangle \sim \langle 1, x^{2^n} + d\rangle \otimes \mathrm{ql}(\varphi) \\ \perp 2\dim\mathrm{ql}(\varphi) \times \langle 0\rangle. \end{aligned}$$

Cancelling the form $2\dim\mathrm{ql}(\varphi) \times \langle 0\rangle$ (Proposition 2.1) and adding $(x^{2^n} + d)R$ to the equation yields

$$R \perp \mathrm{ql}(\varphi) \perp (x^{2^n} + d)\mathrm{ql}(\varphi) \sim (x^{2^n} + d)R \perp \mathrm{ql}(\varphi) \perp (x^{2^n} + d)\mathrm{ql}(\varphi).$$

Since $\mathrm{ql}(\varphi) \perp (x^{2^n} + d)\mathrm{ql}(\varphi) \simeq \mathrm{ql}(\varphi) \perp d\mathrm{ql}(\varphi)$ (because $\langle 1, x^{2^n} + d\rangle \simeq \langle 1, d\rangle$), and the forms on both sides have the same dimension, we deduce

$$\varphi \perp d\mathrm{ql}(\varphi) \simeq (x^{2^n} + d)R \perp \mathrm{ql}(\varphi) \perp d\mathrm{ql}(\varphi).$$

Since $x^{2^n} + d$ is a norm of $\mathrm{ql}(\varphi) \perp d\mathrm{ql}(\varphi)$, we get $\varphi \perp d\mathrm{ql}(\varphi) \simeq (x^{2^n} + d)(\varphi \perp d\mathrm{ql}(\varphi))$, as desired. □

We obtain the following proposition which is a particular case of the implication (1) $\Longrightarrow$ (2) of Theorem 1.1.

Proposition 3.4. *Let $p = x^{2^n} + d \in F[x]$ be an irreducible polynomial and φ a nondefective semisingular quadratic form over F which is quasi-hyperbolic over $F(p)$. Then, p is a norm of φ over $F(x)$.*

Proof. Without loss of generality, we may suppose that φ is anisotropic. Suppose that $\varphi = R \perp \mathrm{ql}(\varphi)$ is quasi-hyperbolic over $F(p)$. In particular, by Proposition 2.2(1) $\mathrm{ql}(\varphi)$ is quasi-hyperbolic over $F(p)$. By statement $(2.c)$ of Proposition 2.2, $\mathrm{ql}(\varphi) \simeq p\mathrm{ql}(\varphi)$ over $F(x)$.

Moreover, by statement $(2.d)$ of Proposition 2.2, we get $\mathrm{ql}(\varphi) \simeq d\mathrm{ql}(\varphi)$. Thus

$$\mathrm{ql}(\varphi) \perp d\mathrm{ql}(\varphi) \simeq \mathrm{ql}(\varphi) \perp \mathrm{ql}(\varphi) \simeq \mathrm{ql}(\varphi) \perp \dim \mathrm{ql}(\varphi) \times \langle 0 \rangle.$$

By statement $(2.b)$ of Proposition 2.2, we get $i_W(\varphi_{F(p)}) = \frac{\dim \varphi - \dim \mathrm{ql}(\varphi)}{2}$. Hence, Proposition 3.3 implies that

$$R \perp \mathrm{ql}(\varphi) \perp d\mathrm{ql}(\varphi) \simeq p(R \perp \mathrm{ql}(\varphi) \perp d\mathrm{ql}(\varphi)).$$

Consequently, we have

$$R \perp \mathrm{ql}(\varphi) \perp \dim \mathrm{ql}(\varphi) \times \langle 0 \rangle \simeq p(R \perp \mathrm{ql}(\varphi)) \perp \dim \mathrm{ql}(\varphi) \times \langle 0 \rangle.$$

Cancelling the form $\dim \mathrm{ql}(\varphi) \times \langle 0 \rangle$ yields $R \perp \mathrm{ql}(\varphi) \simeq p(R \perp \mathrm{ql}(\varphi))$. □

4. Proof of Theorem 1.1 in one variable

We will now prove Theorem 1.1 in this section in the case where the polynomial p is in one variable.

Theorem 4.1. *Let φ be a nondefective semisingular quadratic form of dimension ≥ 3 over F, and let $p \in F[x]$ be a normed irreducible polynomial. If φ is quasi-hyperbolic over $F(p)$, then p is a norm of $\varphi_{F(x)}$.*

Let φ and p be as in Theorem 4.1. Since φ is quasi-hyperbolic over $F(p)$, we get by Proposition 2.2 that p is inseparable and $\mathrm{ql}(\varphi)$ is quasi-hyperbolic over $F(p)$.

Obviously, $p = q(x^{2^m})$ for some $q(x) \in F[x]$ irreducible and separable and some $m \geq 1$. Let us write $F(p) = F(\alpha)$, where α is a root of $p(x)$ in an algebraic extension of F. Let $n = \deg q$, $\beta = \alpha^{2^m}$ and $S = F(\beta)$ which is a separable extension of F. Clearly, $S(x) = F(x)(x^{2^m} + \beta)$ and $q(x^{2^m} + y) \in F(x)[y]$ is the minimal polynomial of $x^{2^m} + \beta$ over $F(x)$.

Let $s : S(x) \to F(x)$ be the $F(x)$-linear map given by:

$$1 \mapsto 1 \text{ and } (x^{2^m} + \beta)^i \mapsto 0$$

for all $1 \leq i \leq \deg q - 1$. We prove the following result:

Proposition 4.2. *We keep the same notations as before. For any nonsingular F-quadratic form R, we have in $W_q(F(x))$:*

$$s_*(R) = \begin{cases} R & \textit{if } \deg q \textit{ is odd,} \\ \langle 1, p \rangle_b \otimes R & \textit{if } \deg q \textit{ is even.} \end{cases}$$

$$s_*((x^{2^m} + \beta)R) = \begin{cases} pR & \textit{if } \deg q \textit{ is odd,} \\ 0 & \textit{if } \deg q \textit{ is even.} \end{cases}$$

Proof. Since $q(x^{2^m} + y) \in F(x)[y]$ is the minimal polynomial of $x^{2^m} + \beta$ over $S(x)$, it follows that $N_{S(x)/F(x)}(x^{2^m} + \beta) = q(x^{2^m}) = p$. Then, the proposition follows from Corollary 2.5. □

For the case of totally singular forms we give the following proposition:

Proposition 4.3. *We keep the same notations as before. For any totally singular quadratic form over F having p as a norm, we have*

$$s_*(\psi) \simeq \psi \perp (n-1) \dim \psi \times \langle 0 \rangle.$$

Proof. Put $q(x) = a_0 + a_1x + \cdots + a_{n-1}x^{n-1} + x^n$. Let $\delta \in F$ and

$$u = \epsilon_0 + \epsilon_1(x^{2^m} + \beta) + \cdots + \epsilon_{n-1}(x^{2^m} + \beta)^{n-1} \in S(x) = F(x)(x^{2^m} + \beta),$$

where $\epsilon_i \in F(x)$. We have

$$s_*(\langle\delta\rangle)(u) = s\big(\delta(\epsilon_0^2 + \epsilon_1^2(x^{2^m} + \beta)^2 + \cdots + \epsilon_{n-1}^2(x^{2^m} + \beta)^{2(n-1)})\big).$$

Since $q(x^{2^m} + y) \in F(x)[y]$ is the minimal polynomial of $x^{2^m} + \beta$ over $F(x)$, it is clear that $(x^{2^m} + \beta)^k \in \bigoplus_{0\leq i\leq n-1} L(x^{2^m} + \beta)^i$ for any $k \geq 0$, where $L = F^2[a_0, \cdots, a_{n-1}, x^{2^m}]$, and thus $s\big(x^{2^m} + \beta)^k\big) \in L$ for any $k \geq 0$. Consequently, $s_*(\langle\delta\rangle)(u) = \delta\epsilon_0^2 + \delta\epsilon_1^2 c_1 + \cdots + \delta\epsilon_{n-1}^2 c_{n-1}$ for suitable $c_1, \ldots, c_{n-1} \in L$. So in terms of isometry it means that

$$s_*(\langle\delta\rangle) \simeq \delta\langle 1, c_1, \ldots, c_{n-1}\rangle.$$

Now write $\psi \simeq \langle\delta_1, \ldots, \delta_r\rangle$ and using the fact that transfer is compatible with the orthogonal sum, we get $s_*(\psi) \simeq \psi \perp c_1\psi \perp \ldots \perp c_{n-1}\psi$. Now statement $(2.d)$ of Proposition 2.2 implies that $c_i\psi \simeq \psi$ as p and x^{2^m} are norms of ψ. Hence, we get $s_*(\psi) \simeq \psi \perp (n-1)\dim\psi \times \langle 0\rangle$. □

Proof of Theorem 4.1. We keep the same notations and hypotheses as before. We may suppose that φ is anisotropic. Note that $\mathrm{ql}(\varphi)$ stays anisotropic over S as the extension S/F is separable [8, Lemma 2.8]. Extending φ to S and using the uniqueness of the quasilinear part, there exists a nonsingular form R_0 over S such that $\varphi_S \simeq R_0 \perp \mathrm{ql}(\varphi)_S \perp i \times [0,0]$ and $R_0 \perp \mathrm{ql}(\varphi)_S = (\varphi_S)_{an}$. Since φ is quasi-hyperbolic over $F(p) = F(\alpha)$, it follows that $(\varphi_S)_{an}$ is quasi-hyperbolic over $F(\alpha)$. Also, $F(\alpha) = S(\alpha)$ and α is purely inseparable over S with minimal polynomial $x^{2^m} + \beta$. Thus, $x^{2^m} + \beta$ is a norm of $(\varphi_S)_{an}$ (we use Proposition 3.4 if $(\varphi_S)_{an}$ is semisingular, and Proposition 2.2(2.c) if $(\varphi_S)_{an}$ is totally singular). Since $\varphi_S \sim (\varphi_S)_{an}$, it follows that

$$\varphi \simeq (x^{2^m} + \beta)\varphi \text{ over } S(x). \tag{4.1}$$

In particular, $\mathrm{ql}(\varphi)_{S(x)} \simeq (x^{2^m} + \beta)\mathrm{ql}(\varphi)_{S(x)}$. To descent the equation (4.1) over $F(x)$, we will use Scharlau's transfer related to the $F(x)$-linear map $s : F(x)(x^{2^m} + \beta) \to F(x)$ given by:

$$1 \mapsto 1 \text{ and } (x^{2^m} + \beta)^i \mapsto 0$$

for all $1 \leq i \leq n-1$. We recall the previous calculations (Propositions 4.2 and 4.3):

$$s_*(R_{S(x)}) \sim \begin{cases} R & \text{if } n \text{ is odd,} \\ \langle 1, p\rangle_b \otimes R & \text{if } n \text{ is even.} \end{cases}$$

$$s_*((x^{2^m} + \beta)R_{S(x)}) \sim \begin{cases} pR & \text{if } n \text{ is odd,} \\ 0 & \text{if } n \text{ is even.} \end{cases}$$

$$s_*(\mathrm{ql}(\varphi)_{S(x)}) \simeq \mathrm{ql}(\varphi) \perp (n-1)\dim\mathrm{ql}(\varphi) \times \langle 0\rangle,$$

Assume that n is odd. Applying the transfer map s_* to the equation (4.1), we obtain:

$$R \perp \mathrm{ql}(\varphi) \perp (n-1)\dim\mathrm{ql}(\varphi) \times \langle 0\rangle \sim pR \perp \mathrm{ql}(\varphi) \perp (n-1)\dim\mathrm{ql}(\varphi)\times \langle 0\rangle. \tag{4.2}$$

Likewise when n is even, we apply the transfer map s_* to the equation (4.1) to get:

$$\langle 1, p\rangle_b \otimes R \perp \mathrm{ql}(\varphi) \perp (n-1)\dim\mathrm{ql}(\varphi) \times \langle 0\rangle \sim \mathrm{ql}(\varphi) \perp (n-1)\dim\mathrm{ql}(\varphi)\times \langle 0\rangle. \tag{4.3}$$

Note that adding pR to both sides of the equation (4.3) gives us the equation (4.2). Now cancelling the form $(n-1)\dim\mathrm{ql}(\varphi) \times \langle 0\rangle$ in the equation (4.2) (Proposition 2.1(2)), and using the fact that $\mathrm{ql}(\varphi) \simeq p\mathrm{ql}(\varphi)$ because $\mathrm{ql}(\varphi)_{F(p)}$ is quasi-hyperbolic (Proposition 2.2(2.c)), we get $\varphi \simeq p\varphi$ over $F(x)$. □

5. Proof of Proposition 1.2

For the proof of Proposition 1.2, we need some preparatory results. First, we mention a lemma to be used in **Step 1** below.

Lemma 5.1. *(*[9, Lemma 2.4]*) Let $p \in F[x_1, \dots, x_n]$ be an irreducible polynomial, and let $\varphi \simeq R \perp \mathrm{ql}(\varphi)$ be an anisotropic quadratic form such that $\dim R > 0$ and $R_{F(p)}$ is not hyperbolic. Then, p stays irreducible over $F(\varphi)$.*

5.1. **Some results on places.** Let K and L be fields. We take $L^\infty = L \cup \{\infty\}$ with the rules:

$$x + \infty = \infty \text{ for } x \in L,$$
$$x\infty = \infty \text{ for } x \in L^*,$$
$$\frac{1}{\infty} = 0, \frac{1}{0} = \infty, \infty\infty = \infty,$$

and $\infty + \infty$, $0 \times \infty$ are not defined.

A place from K to L is a "homomorphism" $\lambda : K \to L^\infty$ satisfying: $\lambda(x+y) = \lambda(x) + \lambda(y)$ and $\lambda(xy) = \lambda(x)\lambda(y)$, whenever the right hand sides are defined (we admit the trivial places $K \hookrightarrow L$).

If K and L are extensions of F and $\lambda(x) = x$ for all $x \in F$, then we say that λ is an F-place. One attaches to λ its ring $R_\lambda := \{x \in K \mid \lambda(x) \neq \infty\}$. This is a valuation ring whose field of fractions is K and maximal ideal is $\mathfrak{m}_\lambda = \{x \in K \mid \lambda(x) = 0\}$. Clearly, the residue field $R_\lambda/\mathfrak{m}_\lambda$ can be identified with a subfield of L. We refer to [10, Appendix, Chapter 3] and [2] for an overview on places and their connection with valuations.

A result that we will use in the sequel is due to Knebusch.

Lemma 5.2. [6, Lemma 2.8] *Let M and N be free quadratic modules over R_λ such that $M/\mathfrak{m}_\lambda M$ and $N/\mathfrak{m}_\lambda N$ are non-degenerate. Assume that $N \otimes K \simeq M \otimes K$. Then, $N/\mathfrak{m}_\lambda N \simeq M/\mathfrak{m}_\lambda M$.*

Using this result we prove a substitution principle for semisingular quadratic forms:

Proposition 5.3. *Let φ be a nondefective semisingular form over F, and let $p \in F[x_1, \dots, x_n]$ be a norm of φ. Let $c_1, \dots, c_k \in F$ be such that the polynomial $q := p(c_1, \dots, c_k, x_{k+1}, \dots, x_n)$ is nonzero, $1 \leq k \leq n$. Then, q is a norm of φ over $F(x_{k+1}, \dots, x_n)$.*

Proof. We give the proof for $k = 1$ and the rest follows by an obvious induction. Let φ be a nondefective semisingular form over F and $p \in F[x_1, \dots, x_n]$ be a norm of φ. Let $c_1 \in F$ be such that $q_1 := p(c_1, x_2, \dots, x_n)$ is nonzero.

Consider $K = F(x_1, \dots, x_n)$ and $L = F(x_2, \dots, x_n)$ (read $L = F$ if $n = 1$). We fix the F-place $\lambda : K \to L^\infty$ given by: $x_1 \mapsto c_1$ and $x_i \mapsto x_i$ for all $2 \leq i \leq n$. Let M and N be free R_λ-module of rank $\dim \varphi$, and equipped with R_λ-quadratic forms Q and Q' such that $Q \simeq \varphi \otimes R_\lambda$ and $Q' \simeq p\varphi \otimes R_\lambda$. Since $\mathrm{ql}(\varphi)$ is anisotropic over L (because φ is nondefective) and $R_\lambda/\mathfrak{m}_\lambda$ is a subfield of L, it follows that $Q \otimes R_\lambda/\mathfrak{m}_\lambda$ and $Q' \otimes R_\lambda/\mathfrak{m}_\lambda$ are non-degenerate. Moreover $Q_K \simeq pQ'_K$ because $\varphi_K \simeq p\varphi_K$. Hence, Lemma 5.2 implies that $Q \otimes R_\lambda/\mathfrak{m}_\lambda \simeq Q' \otimes R_\lambda/\mathfrak{m}_\lambda$. Extending scalars to L, we get $\varphi_L \simeq q\varphi_L$, as desired. □

5.2. **Proof of Proposition 1.2.** Let $\varphi \simeq R \perp \mathrm{ql}(\varphi)$ be a nondefective semisingular quadratic form over F. Let $f \in F[x_1, \dots, x_n]$ be a norm of φ and p a normed irreducible polynomial that divides f with an odd power. We want to prove that p is a norm of φ. Without loss of generality, we may suppose that φ is anisotropic and p^2 does not divide f, i.e., $f = pg$ where p does not divide g. We proceed by induction on n.

Step 1. The case $n = 1$. We will follow some arguments used in the proofs of [9, Lemma 2.3, Theorem 1.1]. Since f is a norm of φ, we have $R \perp \mathrm{ql}(\varphi) \simeq pg(R \perp \mathrm{ql}(\varphi))$. By the uniqueness of the quasilinear part, we have $\mathrm{ql}(\varphi) \simeq pg(\mathrm{ql}(\varphi))$. By [8, Proposition 1.2], we get $\mathrm{ql}(\varphi) \simeq p\mathrm{ql}(\varphi)$, and thus $\mathrm{ql}(\varphi)_{F(p)}$ is quasi-hyperbolic.

Claim 1. $\mathrm{ql}(\varphi_{F(x)}) \simeq S_{F(x)} \perp pS_{F(x)}$ for a suitable subform S of $\mathrm{ql}(\varphi)$.

By [7, Lemma 2.1], there exists a subform S of $\mathrm{ql}(\varphi)$ such that $(\mathrm{ql}(\varphi)_{F(p)})_{an} \simeq S_{F(p)}$. Let us write $S = \langle c_1, \dots, c_s \rangle$. It suffices to prove that the elements $c_1, \dots, c_s, pc_1, \dots, pc_s$ are $F(x)^2$-linearly independent by [7, Lemma 2.1]. In fact, let $q_1, \dots, q_s, q'_1, \dots, q'_s \in F(x)$, not all zero, be such that

$$\sum_{i=1}^{s} c_i q_i^2 + p \sum_{i=1}^{s} c_i q_i'^2 = 0. \tag{5.1}$$

We may suppose that $q_1, \dots, q_s, q'_1, \dots, q'_s \in F[x]$ and p does not divide all of them. We extend equation (5.1) to $F(p)$ to get $\sum_{i=1}^{s} c_i \bar{q}_i^2 = 0 \in F(p)$. Since $S_{F(p)}$ is anisotropic, it follows that $q_i = r_i p$ for some $r_i \in F[x]$ $(1 \leq i \leq s)$. We substitute $q_i = r_i p$ in equation (5.1), we simplify by p and extend to $F(p)$ to get $\sum_{i=1}^{s} c_i \overline{q'_i}^2 = 0 \in F(p)$. Again, the anisotropy of $S_{F(p)}$ implies that p divides $q'_1, \dots, q'_s$, a contradiction to the choice of $q_1, \dots, q_s, q'_1, \dots, q'_s$. Hence the claim.

Claim 2. $i_W(\varphi_{F(p)}) \geq 1$.

Let us assume that $i_W(\varphi_{F(p)}) = 0$ and let $r = \dim R$. The previous claim gives us the isometry

$$pg\varphi \simeq R \perp S \perp pS \text{ over } F(x).$$

Without loss of generality, we assume that $1 \in D_F(\varphi)$ and thus $f \in D_{F(x)}(\varphi)$. Hence, there exists $u \in F[x]^r$, $v, v' \in F[x]^s$ and $q \in F[x]$ such that

$$pgq^2 = R(u) + S(v) + pS(v'). \tag{5.2}$$

We may suppose that q and the polynomials composing u, v and v' are coprime. We extend equation (5.2) to $F(p)$ to get $R(\bar{u}) + S(\bar{v}) = 0$. Using $i_W(\varphi_{F(p)}) = 0$ and anisotropy of S, it follows that $u = pu_1$ and $v = pv_1$ for some $u_1 \in F[x]^r$ and $v_1 \in F[x]^s$. Substituting $u = pu_1$ and $v = pv_1$ in equation (5.2) and simplifying by p, we get $S(v') = gq^2 + pl$ for some $l \in F[x]$. In particular, we can say that $\mathrm{ql}(\varphi)$ represents $gq^2 + pl$ over $F[x]$. We have $\mathrm{ql}(\varphi) \simeq pg\mathrm{ql}(\varphi)$, therefore $pg\mathrm{ql}(\varphi)$ represents $gq^2 + pl$ over $F[x]$, i.e.,

$$gq^2 + pl = pgS(q_1) + p^2 gS(q_2) \tag{5.3}$$

for some $q_1, q_2 \in F[x]^s$. We extend equation (5.3) to $F(p)$ to get $g\bar{q}^2 = 0$, i.e., $q = pq'$ for some $q' \in F[x]$. We substitute this in equation (5.2), simplify by p and extend to $F(p)$ to get $S(\bar{v}') = 0$. Since S is anisotropic over $F(p)$, we get $v' = pv'_1$ for some $v'_1 \in F[x]^s$. This is a contradiction to the hypothesis that q and the polynomials composing u, v and v' are coprime. Thus, our assumption that $i_W(\varphi_{F(p)}) = 0$ is wrong. We now have $i_W\big(\varphi_{F(p)}\big) \geq 1$.

Claim 3. $i_W\big(\varphi_{F(p)}\big) = \frac{\dim R}{2}$.
To prove the claim we proceed by induction on $\dim R$. If $\dim R = 2$, then we are done by Claim 2. Suppose that $\dim R > 2$ and the claim is true for any nondefective semisingular form φ' that has f as a norm and whose regular part is of dimension $< \dim R$.
Let $L = F(\varphi)$ and put $\varphi_L \simeq R' \perp i \times \mathbb{H} \perp \mathrm{ql}(\varphi)$, where $i = i_W(\varphi_L)$. We treat two cases:

(a) If $R_{F(p)}$ is hyperbolic, then $i_W(\varphi_{F(p)}) = \frac{\dim R}{2}$, and we are done.
(b) If $R_{F(p)}$ is not hyperbolic. Lemma 5.1 implies that p remains irreducible over L. Since f_L is also a norm of $R' \perp \mathrm{ql}(\varphi)_L$, it follows by induction hypothesis that $i_W((R' \perp \mathrm{ql}(\varphi))_{L(p)}) = \frac{\dim R'}{2}$.

Hence, we get $i_W(\varphi_{L(p)}) = \frac{\dim R}{2}$. Moreover, the extension $F(p)(\varphi)/F(p)$ is purely transcendental since $i_W(\varphi_{F(p)}) \geq 1$ (Claim 2). As $L(p) = F(p)(\varphi)$, we conclude from $i_W(\varphi_{L(p)}) = \frac{\dim R}{2}$ that $i_W(\varphi_{F(p)}) = \frac{\dim R}{2}$, as desired.
In conclusion of Step 1, we got $\mathrm{ql}(\varphi)_{F(p)}$ quasi-hyperbolic and $i_W(\varphi_{F(p)}) = \frac{\dim R}{2}$, which implies that $\varphi_{F(p)}$ is quasi-hyperbolic. By Theorem 4.1, we conclude that p is a norm of φ.

Step 2. Suppose $n \geq 2$ and the proposition is true for $n-1$. We will use an induction argument due to Knebusch [6, Page 296-297]. Set $x' = (x_2, \ldots, x_n)$, $x = (x_1, x')$ and $L = F(x_2, \ldots, x_n)$. Let r be the degree of p considered as a polynomial in $L[x_1]$ and $\zeta \in F[x_2, \ldots, x_n]$ be the highest coefficient of $p \in L[x_1]$.

(1) Suppose that F is infinite:

- If $r = 0$, i.e., p is a constant polynomial in $L[x_1]$. We write $f = p(x')g(x) \in F(x_1, \ldots, x_n)$. Since p^2 does not divide f, then p does not divide all coefficients of $g \in L[x_1]$. Since F is infinite, there exists $c \in F$ such that $p(x')$ does not divide $g(c, x')$ in $F[x']$. By Proposition 5.3, $p(x')g(c, x')$ is a norm φ_L. By induction hypothesis $p(x')$ is a norm of φ_L, and thus it is also a norm of φ_K.
- If $r > 0$. Let $p' = \zeta^{-1}p$ which is a normed polynomial in $L[x_1]$. We will first verify that p'^2 does not divide f. Assume that $p'^2 | f$, then $p' | \zeta g$ and thus $p | \zeta^2 g$. This is not possible since p is an irreducible polynomial which does not divide g. Hence, $p'^2 \nmid f$. We have $\varphi \simeq f\varphi \simeq pg\varphi \simeq \zeta^{-1}p'g\varphi$. Using Step 1, we get $\varphi_{L(x_1)} \simeq p'\varphi_{L(x_1)}$, i.e.,
$$p\varphi_{L(x_1)} \simeq \zeta\varphi_{L(x_1)}.$$
We claim that ζ is a norm of φ_L. Let us take h any normed irreducible divisor of ζ in $F[x_2, \ldots, x_n]$ with odd power, say $\zeta = h\zeta'$. Since p is irreducible, the polynomial h does not divide all coefficients of $p \in L[x_1]$. Since F is infinite, there exists $c \in F$ such that h does not divide $p(c, x')$. By Proposition 5.3, we have the isometry $p(c, x')\varphi_L \simeq \zeta\varphi_L$. Hence, $\zeta p(c, x')$ is a norm of φ_L, and by induction hypothesis h is a norm of φ_L.
Since ζ is normed and any normed irreducible factor of it is a norm of φ_L, we deduce that ζ is a norm of φ_L, and thus p is a norm of $\varphi_{L(x_1)}$.

(2) Suppose that F is finite. We change F by $F(t)$ for some variable t over F. Hence, over $F(t)$ we are in condition (1), and thus p is a norm of $\varphi_{F(t)(x_1,\ldots,x_n)}$. Now applying Proposition 5.3 and substituting $t = 0$, we get $\varphi \simeq p\varphi$ over $F(x_1, \ldots, x_n)$. $\square$

6. Proof of Theorem 1.1 in many variables

Let $p \in F[x_1, x_2, \ldots, x_n]$ be a normed irreducible polynomial, $L = F(x_2, \ldots, x_n)$ and let ζ be the highest coefficient of p considered as a polynomial of $L[x_1]$. Let φ be a nondefective semisingular quadratic form of dimension ≥ 3 over F which is quasi-hyperbolic over $F(p) = L(p)$. By Theorem 4.1 the polynomial $\zeta^{-1}p \in L[x_1]$ is a norm of $\varphi_{L(x_1)}$, or, equivalently ζp is a norm of $\varphi_{L(x_1)}$. By Proposition 1.2, p is a norm of φ.

Conversely, if p is a norm of φ, then $\varphi_{F(p)}$ is quasi-hyperbolic by [9, Theorem 1.1]. $\square$

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